Approach to overcome difficulties when solving FPK at the first timestep

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1 Existing Problem

In nonlinear spectral stochastic finite element scheme (SSFEM), a FPK equation, also called advection-diffusion equation, is utilized to compute the probability density function (PDF) of stress.

The initial condition of the stress should be assumed as a dirac-delta function, or a standard normal function for ease of implementation in following time steps.

However, the advection and diffusion coefficient, denoted as N_1 and N_2 respectively, can be very large or very small depending on the loading time history, even at the first time step. The problem is that it is hard to solve the FPK with very large N_1 or N_2 since the initially assigned stress domain is too small.

It is easy to manage large N_1, since the advection distance can be directly calculated as $N_1 * dt$. However, it is hard to get a quantitative estimate of the diffusion effect with large N_2. Apparently, a large stress domain is required to capture the diffusive effect with a large N_2. But, the initial assumed standard Gaussian PDF would look too small and the mesh should be very fine to capture the shape of the PDF.

Traditionally, the author tried to manually change the stress domain size and increase the standard deviation of the initial PDF. The procedure is tedious and can not be generalized to other cases.

Table 1: Comparison of Finite Difference and Eq. 1 Apparoximation

Diffusion coeff.	Finite Difference	Eq. (1) Approximation
1	1.0001	0.01
10	1.001	0.032
1e2	1.01	0.14
1e3	1.095	0.4472
1e4	1.73	1.44
1e5	4.58	4.47
1e6	14.17	14.14
1e7	44.73	44.72

2 Proposed solution

Here, the author just find out an approximate and straightforward way to overcome this difficulty. The finding is from Eistein's paper in 1905 on Brownian motion. The paper finds that the standard deviation of the distance from the particle to its original place is :

$$\sqrt{2N_2t} \tag{1}$$

We can use this finding to approximate a solution of the FPK without solving it. One assumption is that the material is still elastic at the first time step. This is usually the case with nonlinear structural analysis.

If we assume the initial condition is a dirac-delta function, the approximated solution at the first time step is exact. If we assume the initial condition to be a standard normal function, the approximated solution is still acceptable when $N_2 > 1e5$. The comparison of finite difference solution and Eq. 1 approximation.

From Table 1, it can be observed that the approximation is in good agreement with finite difference solution when $N_2 > 1e5$. As we said before, we have difficulties with large N_2. Now Eq. 1 could give us good approximation with large N_2. On the other hand, we could solve the FPK equation with small N_2 since it's very easy to assign a good stress domain (because the standard deviation of the solution is close to initial assumed PDF).

3 Summary

An approximation approach is found to overcome the difficulties to solve FPK with very large N_2 at the first time step. The approximation approach is exact for dirac-delta initial condition, while it is also good enough for large N_2 ($>$1e5) with standard Gaussian initial condition. Note that the assumption is that the material remains elastic after first time step (usually the case).